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Natural convection from a vertical cylinder at very large Prandtl numbers

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SUMMARY
The natural convection from a vertical cylinder, is determined in the limiting case of very large Prandtl number, the Grashof number remaining finite.

1. Introduction

This paper studies the natural convection flow on a vertical cylinder when the Prandtl number P is very large. Little work has been done in this field apart from the experiments of Libby [1] and a numerical calculation made by Fujii and Uehara [2] for the case $P = 100$.

The method adopted is to split the flow into a thin layer close to the surface of the cylinder (where the temperature varies), surrounded by a much thicker layer where the velocity is reduced to zero. The solution is determined, in the inner region in terms of a parameter which is roughly equal to the ratio of the thickness of this layer to the radius of the cylinder; it is valid up to a vertical height at which this parameter is about unity. The basic properties of the flow are evaluated; the heat transfer coefficient is shown to be in qualitative agreement with Libby [1].

2. Equations of motion

Let cylindrical coordinates (x, r) be taken whose axis is the vertical centre line of the cylinder and whose origin is at the centre of the base of the cylinder. Let (u, v) be the corresponding velocity components.

Then the boundary layer equations are

$$\frac{\partial}{\partial x}(ur) + \frac{\partial}{\partial r}(vr) = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + g\beta_1\theta(T_w - T_\infty), \tag{2}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial r} = \frac{\nu}{rP}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right), \tag{3}$$

where ν is the kinematic viscosity, g is the acceleration due to gravity, β_1 is the coefficient of volumetric expansion, P is the Prandtl number and

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \tag{4}$$

where T is the temperature of the fluid and T_∞, T_w are the constant temperatures of the surrounding fluid and cylinder respectively.

Equation (1) may be eliminated by introducing a stream function ψ such that

$$ur = \frac{\partial\psi}{\partial r}, \quad vr = -\frac{\partial\psi}{\partial x}. \tag{5}$$

The boundary conditions are: on $r = a$, the surface of the cylinder, $u = v = 0$, $\theta = 1$, and at large radial distances u, θ tend to zero.

3. Flow in the inner temperature layer

The problem of natural convection from a vertical flat plate at large values of P has been discussed by Stewartson and Jones [3] and independently by Kuiken [4]. These reports show that the flow over a flat plate consists of two regions, namely, a thin temperature region, where buoyancy is roughly balanced by viscosity and a thick momentum layer, where the temperature is approximately constant. The thicknesses of these regions were found to be of respective orders

$$\left(\frac{4}{GP}\right)^{\frac{1}{4}} x \text{ and } \left(\frac{4P}{G}\right)^{\frac{1}{4}} x,$$

where $G \equiv$ Grashof number

$$\equiv \frac{g\beta_1(T_w - T_\infty)x^3}{\nu^2}. \tag{6}$$

Thus in the limit as $P \to \infty$ with G finite, the temperature layer becomes vanishingly thin while the momentum region becomes infinitely thick. This suggests, in the cylindrical case, that at finite values of x for sufficiently large P the thickness of the temperature layer will be much smaller than the radius of the cylinder.

Thus the appropriate variables inside the inner layer are:

$$X = \left(\frac{64}{GP}\right)^{\frac{1}{4}} \frac{x}{a}, \quad Y = \frac{r^2 - a^2}{a^2 X},$$
$$\psi = 2\sqrt{2}\, \nu a \left(\frac{G}{P^3}\right)^{\frac{1}{4}} f(X, Y); \tag{7}$$

it should be noted that X gives the order of magnitude of the ratio of the thickness of the temperature layer to the radius of the cylinder. This change of variable is an adaption of that used by Sparrow and Gregg [5] in their discussion of this problem at moderate Prandtl numbers.

From (7) it is readily shown that

$$u = \frac{2\nu}{x}\left(\frac{G}{P}\right)^{\frac{1}{2}} \frac{\partial f}{\partial Y}. \tag{8}$$

The variables (7) and (8) reduce to Kuiken's inner flat plate variables in the limit as $a \to \infty$.
When (7), (8) are inserted in (2), (3) the following equations are obtained

$$(1 + XY)\frac{\partial^3 f}{\partial Y^3} + X\frac{\partial^2 f}{\partial Y^2} + \theta + \frac{1}{P}\left[-X\left\{\frac{\partial^2 f}{\partial X \partial Y}\frac{\partial f}{\partial Y} - \frac{\partial^2 f}{\partial Y^2}\frac{\partial f}{\partial X}\right\} + 3f\frac{\partial^2 f}{\partial Y^2} - 2\left(\frac{\partial f}{\partial Y}\right)^2\right] = 0 \tag{9}$$

and

$$(1 + XY)\frac{\partial^2 \theta}{\partial Y^2} + X\frac{\partial \theta}{\partial Y} - X\left(\frac{\partial f}{\partial Y}\frac{\partial \theta}{\partial X} - \frac{\partial \theta}{\partial Y}\frac{\partial f}{\partial X}\right) + 3f\frac{\partial \theta}{\partial Y} = 0. \tag{10}$$

The boundary conditions on f, θ at $Y = 0$ are that

$$f = \frac{\partial f}{\partial Y} = 0, \quad \theta = 1.$$

For very large values of P, (9), (10) reduce to

$$\frac{\partial^3 f}{\partial Y^3} + \theta = 0, \tag{11}$$

$$\frac{\partial^2 \theta}{\partial Y^2} + 3f\frac{\partial \theta}{\partial Y} = 0, \tag{12}$$

with an error of order X *i.e.* of order $P^{-\frac{1}{4}}$. Equations (11), (12) have been solved in [3] and [4]. Kuiken gives the following results for the solutions f_0, θ_0 of (11), (12):

$$\left.\begin{aligned} &\frac{\partial^2 f_0}{\partial Y^2} = .825, \quad \frac{\partial \theta_0}{\partial Y} = -.711 \text{ at } Y = 0 \\ &\text{and} \\ &f_0(Y) = a_0 + a_1 Y, \text{ with } a_0 = -.261,\ a_1 = .511 \\ &\text{and} \\ &\theta_0 \to 0 \end{aligned}\right\} \tag{13}$$

in the limit as $Y \to \infty$.

To summarise, when P is large, and the Grashof number is finite, the inner temperature layer forms a thin skin (whose thickness varies as $P^{-\frac{1}{4}}$) on the surface of the cylinder. From (13) it can be shown that the outer surface of this skin moves with velocity $U_0(x)$, where

$$U_0(x) \doteqdot \frac{2\nu a_1}{x}\left(\frac{G}{P}\right)^{\frac{1}{2}} \tag{14}$$

4. Flow in the outer momentum layer

In the outer momentum layer, where the temperature is sensibly constant, the cylinder appears to move with velocity $U_0(x)$. In this section the flow corresponding to such a moving cylinder is developed. (It will be verified in section 5 that this solution does in fact match the inner solution). Now the boundary layer due to a cylinder which moves with constant velocity has been determined by Crane [6]. The method used in (6) suggests the following change of variables when the radius of the cylinder is very much thinner than the boundary layer:

$$\psi = \nu x\, F(\eta, \beta), \qquad \eta = \frac{U_0 r^2}{4\nu x}, \tag{15}$$

$$\beta = \log\left(\frac{4\nu x}{U_0 a^2}\right) = 2\log X + \log P - \log(4a_1)$$

$$= 2\log X + \log P - .716.$$

In terms of (15)

$$\frac{u}{U_0} = \frac{1}{2}\,\frac{\partial F}{\partial \eta}, \quad v = -\frac{\nu}{r}\left[F + \frac{1}{2}\,\frac{\partial F}{\partial \beta} - \tfrac{1}{2}\eta\,\frac{\partial F}{\partial \eta}\right] \tag{16}$$

and equation (2), with $\theta = 0$, reduces to:

$$\frac{\partial}{\partial \eta}\left(\eta\,\frac{\partial^2 F}{\partial \eta^2}\right) + \tfrac{1}{2}F\,\frac{\partial^2 F}{\partial \eta^2} - \frac{1}{4}\left(\frac{\partial F}{\partial \eta}\right)^2 - \frac{1}{4}\,\frac{\partial^2 F}{\partial \beta \partial \eta}\,\frac{\partial F}{\partial \eta} + \frac{1}{4}\,\frac{\partial F}{\partial \beta}\,\frac{\partial^2 F}{\partial \eta^2} = 0. \tag{17}$$

The boundary conditions are: on the surface of the moving cylinder, which may be taken to a good approximation at $r = a$, *i.e.* $\eta = e^{-\beta}$,

$$\frac{\partial F}{\partial \eta} = 2 \ \text{ and } \ F + \frac{1}{2}\,\frac{\partial F}{\partial \beta} = e^{-\beta}, \tag{18}$$

while $\partial F/\partial \eta \to 0$ as $\eta \to \infty$.

Following [6], it will be assumed that, in the limit as $\beta \to \infty$ (which occurs when $P \to \infty$) the terms in $\partial/\partial\beta$ tend to zero. Then, for large values of β equation (17) becomes approximately

$$\frac{\partial}{\partial \eta}\left(\eta\,\frac{\partial^2 F}{\partial \eta^2}\right) + \tfrac{1}{2}F\,\frac{\partial^2 F}{\partial \eta^2} - \frac{1}{4}\left(\frac{\partial F}{\partial \eta}\right)^2 = 0. \tag{19}$$

Equation (19) has the property that if $F_0(\eta)$ is a solution so is $F_0(z)$ where $z = \eta/\gamma$ and γ is any positive function of β. Equation (19) then becomes

$$\frac{d}{dz}\left(z\,\frac{d^2 F_0}{dz^2}\right) + \tfrac{1}{2}F_0\,\frac{d^2 F_0}{dz^2} - \frac{1}{4}\left(\frac{dF_0}{dz}\right)^2 = 0. \tag{20}$$

A standard solution of (20) was calculated subject to the conditions:

$$\frac{dF_0}{dz} \to 0 \text{ as } z \to \infty$$

$$F_0 \to 0 \text{ as } z \to 0.$$

When z is large

$$F_0 = 2(D+1) + \sum_{n=1}^{\infty} \frac{A_n}{Z^{nD}}, \tag{21}$$

where D is a constant to be determined. The first few terms of (21) and its derivatives provide starting values (at a sufficiently large value of z) for the numerical integration of (20) in the direction of decreasing z, see table 2. When z is small it is found that, when $D = .539$,

$$F_0 \sim Az(\log z + B - 1) + O(z^2(\log z)^2),$$

$$\frac{dF_0}{dz} \sim A(\log z + B) + O(z),$$

where $A = -30.90$, $B = 4.61$.

Now when z is small

$$\frac{\partial F}{\partial \eta} = \frac{1}{\gamma}\frac{dF_0}{dz} = \frac{A(\log z + B)}{\gamma} + O(z).$$

It follows that the first part of condition (18) will be satisfied, to within a fractional error of order $e^{-\beta}$, (*i.e.* $P^{-\frac{1}{2}}$) provided

$$\beta = \beta^* - \log \beta^* + B - \log(-\tfrac{1}{2}A), \tag{22}$$

where $\beta^* = -2\gamma/A$.

Thus a first approximation to the outer velocity profile is

$$\frac{u}{U_0} = \frac{1}{\gamma}\frac{dF_0}{dz}.$$

The error due to the neglected terms in $\partial/\partial\beta$ in (17) is readily shown to be of order $(\beta^*)^{-1}$.

This error may be corrected by expanding F in the series

$$F = F_0(z) + \frac{1}{\beta^*}F_1(z) + \ldots \tag{23}$$

The equation for F_1 is found by substituting (23) in (17) and equating terms of order $(\beta^*)^{-1}$. Then

$$z\frac{d^3F_1}{dz^3} + \frac{d^2F_1}{dz^2} + \tfrac{1}{2}F_0\frac{d^2F_1}{dz^2} + \tfrac{1}{2}F_1\frac{d^2F_0}{dz^2} - \frac{1}{2}\frac{dF_0}{dz}\frac{dF_1}{dz} + \frac{1}{4}\left(\frac{dF_0}{dz}\right)^2 = 0. \tag{24}$$

The solution of (24) for which

$$(F_1)_0 = \left(\frac{d^2F_1}{dz^2}\right)_0 = 0, \quad \left(\frac{dF_1}{dz}\right)_\infty = 0$$

is given in table 2; note the values

$$\left(\frac{dF_1}{dz}\right)_0 = \bar{C}_1 = 20.0, \quad F_1(\infty) = .154.$$

Now at $\eta = e^{-\beta}$

$$\frac{u}{U_0} = \frac{1}{2\gamma}[A(-\beta - \log\gamma + B) + \bar{C}_1/\beta^*] = 1.$$

It follows that the definition (22) of β^* must be amended to

$$\begin{aligned}\beta &= \beta^* - \log\beta^* + B - \log(-\tfrac{1}{2}A) + \bar{C}_1/A\beta^* \\ &= \beta^* - \log\beta^* + 1.87 - .65/\beta^*.\end{aligned} \tag{25}$$

To summarise: the outer velocity profile is

$$\frac{u}{U_0} = -\frac{1}{A\beta^*}\left[\frac{dF_0}{dz} + \frac{1}{\beta^*}\frac{dF_1}{dz} + O\left(\frac{1}{\beta^{*2}}\right)\right]. \tag{26}$$

This formula gives u/U_0 to within a few percent when β^* or β is greater than about 5. This will be roughly true for $X > .1$ when $P > 10^4$.

5. Higher approximations to the inner solution

The error in the approximate inner solution (13) has two sources, firstly that (13) is a solution of the truncated equations (11), (12) in which terms of order X have been neglected; and secondly that the outer solution (26) does not exactly match the inner solution (13).

The first source of error may be approximately corrected by writing

$$f = f_0 + Xf_1, \quad \theta = \theta_0 + X\theta_1. \tag{27}$$

When (27) are inserted in (9), (10) and terms of order X equated it is found that

$$\frac{d^3f_1}{dY^3} + \theta_1 + Y\frac{d^3f_0}{dY^3} + \frac{d^2f_0}{dY^2} = 0, \tag{28}$$

$$\frac{d^2\theta_1}{dY^2} + 3f_0\frac{d\theta_1}{dY} - \frac{df_0}{dY}\theta_1 + 4f_1\frac{d\theta_0}{dY} + \frac{d\theta_0}{dY} + Y\frac{d^2\theta_0}{dY^2} = 0. \tag{29}$$

Table (1) gives the solution to (28), (29) which satisfies the conditions:
on $Y = 0$,

$$f_1 = \frac{df_1}{dY} = \theta_1 = 0$$

and at $Y = \infty$,

$$\frac{df_1}{dY} = 0, \quad \theta_1 = 0.$$

To turn to the second source of error. The inner limit of the outer flow (26) is

$$u = \frac{U_0}{2}\frac{\partial F}{\partial \eta}$$
$$= U_0\left[1 - \frac{1}{\beta^*}\log(1 + XY)\right]$$
$$= U_0\left[1 - \frac{XY}{\beta^*} + O(X^2)\right],$$

when the outer solution is expressed in terms of X, Y.

Now the outer limit of the inner flow is

$$u = U_0.$$

It follows that a correction term of order X/β^* must be added to (27), *i.e.*

$$f = f_0 + Xf_1 + \frac{X}{\beta^*}f_{12}, \quad \theta = \theta_0 + X\theta_1 + \frac{X}{\beta^*}\theta_{12}, \tag{28}$$

where

$$\frac{df_{12}}{dY} \sim b_1 - a_1 Y, \ \theta_{12} \sim 0, \text{ for large values of } Y,$$

and at $Y = 0$,

$$f_{12} = \frac{df_{12}}{dY} = \theta_{12} = 0.$$

The solution f_{12} which satisfies these conditions is given in table 1, from this integration it is found that

$$b_1 = .34.$$

It should be noted that the term in b_1 is unmatched in the outer flow; this defect may be corrected by amending the value of $U_0(x)$ to

$$U_0(x) = \frac{2\nu}{x}\left(\frac{G}{P}\right)^{\frac{1}{2}}\left[a_1 + \frac{b_1 X}{\beta^*}\right]. \tag{29}$$

6. Boundary layer properties

In this section some of the overall properties of the flow are brought together.

Maximum velocity $= U_0(x)$

$$= \frac{2\nu}{x}\left(\frac{G}{P}\right)^{\frac{1}{2}}\left[a_1 + \frac{b_1 X}{\beta^*} + O(X^2)\right]. \tag{30}$$

Rate at which mass is entrained/unit length of cylinder

$$= 2\pi \rho v x F(\infty)$$
$$= 2\pi \rho v x \left[F_0(\infty) + \frac{1}{\beta^*} F_1(\infty) + O\left(\frac{1}{\beta^{*2}}\right) \right]$$
$$= 2\pi \rho v x \left[3.078 + \frac{.154}{\beta^*} \right]. \tag{31}$$

Heat transfer, in this case expressed in terms of the Nusselt number N defined by

$$N \equiv -2\pi a \left(\frac{\partial \theta}{\partial r}\right)_{r=a},$$

is most conveniently expressed in terms of the dimensionless number $\bar{N}$ defined by

$$\bar{N} = \frac{xN}{2\pi a (GP)^{\frac{1}{4}}}$$
$$= -\frac{1}{\sqrt{2}} \left[\frac{d\theta_0}{dY} + X \frac{d\theta_1}{dY} + \frac{X}{\beta^*} \frac{d\theta_{12}}{dY} + O(X^2) \right]_{Y=0}$$
$$= .503 + .186 X - .123 \frac{X}{\beta^*}. \tag{32}$$

Note that X, β^* are defined by equations (7), (25) respectively.

7. Discussion

The most important of the above properties is (32) for which it should be noted that the term in X/β^* is negligible for $P > 10^4$ and $X < 1$. It follows that (32) is now sensibly

$$\bar{N} = .503 + .186 X + O(X^2). \tag{33}$$

It is interesting to compare (33) with the numerical calculations of Fujii and Uehara [2], for $P = 100$, whose results when expressed in the notation of this paper give

$$\bar{N} = .486 + .051 X - .003 X^2. \tag{34}$$

While the agreement between (33), (34) cannot be expected to be good, nevertheless (34) indicates that the error in (33) should be less than one per cent even when $X = 1$.

The experiments of Libby [1] (reported in a review article by Soehngen [7]) were performed on a cylinder of radius 1.85 cm at a vertical height of 7 cm in the ranges

$$10^4 < P < 10^6, \quad 10^4 < PG < 10^8.$$

This gives a range of values of X from .1 to 1 and of β from 4 to about 12, in which (33) is valid. Libby's results show a linear dependence of N on $(GP)^{\frac{1}{4}}$ giving a value of $\bar{N} = .47$. It is surprising that no curvature effects were observed in view of the range of values of X. However the experimental values for wall temperature gradient were obtained from the temperature profile. Now consideration of table 1, shows that the average value of θ_1 in the range $0 \leqq Y \leqq .75$ (which corresponds roughly to the linear part of the temperature profile), is about .02. It follows that the average effect of curvature on the temperature profile in this range is at most about 2% (when $X = 1$). Thus curvature will have no sensible effect on values of wall temperature gradient derived from points on the experimental temperature profile.

TABLE 1

Y	f_0	$\frac{df_0}{dY}$	θ_0	f_1	$\frac{df_1}{dY}$	θ_1	f_{12}	$\frac{df_{12}}{dY}$	θ_{12}
0	0	0	1.000	0	0	0	0	0	0
.25	.023	.177	.822	0	.01	−.04	− .01	− .07	.04
.50	.084	.302	.649	0	0	−.05	− .03	− .14	.08
.75	.171	.387	.485	0	−.01	−.01	− .08	− .21	.12
1.00	.275	.441	.342	0	−.02	.04	− .14	− .29	.14
1.25	.390	.474	.225	−.01	−.03	.09	− .22	− .38	.15
1.50	.511	.492	.138	−.02	−.02	.11	− .33	− .48	.13
1.75	.635	.502	.078	−.02	−.01	.12	− .46	− .59	.11
2.00	.761	.507	.041	−.02	−.01	.10	− .62	− .70	.08
2.25	.890	.509	.020	−.02	0	.07	− .81	− .82	.05
2.50	1.016	.510	.009	−.02	0	.04	−1.04	− .94	.03
2.75	1.143	.510	.003	−.02	0	.02	−1.29	−1.07	.02
3.00	1.271	.511	.001	−.02	0	.01	−1.57	−1.19	.01
3.25	1.399	.511	0	−.02	0	.01	−1.88	−1.32	0
3.50	1.526	.511	0	−.02	0	0	−2.23	−1.45	0

TABLE 2

log z	F_0	$\frac{dF_0}{dz}$	$\frac{d^2F_0}{dz^2}$	F_1	$\frac{dF_1}{dz}$	$\frac{d^2F_1}{dz^2}$
−12	.001	227	−30.8	0	20.0	− .082
−10	.009	166	−30.3	.001	19.5	− .416
− 8	.046	107	−28.5	.006	17.8	−1.49
− 6	.199	54.4	−22.9	.038	12.8	−3.51
− 4	.659	18.4	−12.5	.160	4.96	−3.64
− 2	1.49	3.36	− 3.42	.343	.50	− .85
0	2.30	.326	−.422	.378	−.026	− .003
2	2.77	.021	−.030	.297	−.006	.007
4	2.97	.001	−.002	.224	−.0005	.0006
6	3.04	.0001	−.0001	.184	−.00003	.00005
8	3.07	.000002	−.000004	.167	−.000005	.000003

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